Epicyclic drifting in anisotropic excitable media with multiple inhomogeneities

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Abstract

Spirals have been studied from a dynamical system perspective starting with Barkley's seminal papers linking a wide class of spiral wave dynamics to the Euclidean symmetry of the excitable media in which they are observed. However, in order to explain certain non-Euclidean phenomena, such as anchoring and epicyclic drifting, LeBlanc and Wulff introduced a single translational symmetry-breaking perturbation to the center bundle equation and showed that rotating waves may be attracted to a non-trivial solution manifold and travel epicyclically around the perturbation center.

In this paper, we continue the (model-independent) investigation of the effects of inhomogeneities on spiral wave dynamics by studying epicyclic drifting in the presence of: a) n simultaneous translational symmetry-breaking terms, with $n > 1$, and b) a combination of a single rotational symmetry-breaking term and a single translational symmetry-breaking term. These types of forced Euclidean symmetry-breaking provide a much more realistic model of certain excitable media such as cardiac tissue. However, the main theoretical tool used by LeBlanc and Wulff can only be applied to their particular perturbation: we show how an averaging theorem of Hale can be modified to analyze our two more general scenarios and state the conditions under which epicyclic drifting takes place in the general case. In the process, we recover LeBlanc and Wulff's specific result. Finally, we illustrate our results with the help of a simple numerical simulation of a modified bidomain model.

Keywords: symmetry-breaking, integral manifold, epicycle, spiral wave, excitable media, averaging, center bundle equation, center manifold reduction theorem.

1 Introduction

Spiral are found in numerous excitable media [1, 2, 4, 9, 13, 16, 21, 22, 25, 27, 35, 37, 39, 40] and they give rise to beautiful imagery. While this in itself might yield enough interest to study them, there is also (at least) one serious reason to do so: spiral waves have been linked to cardiac arrhythmias (to disruptions of the heart's normal electrical cycle) [9, 17, 36, 37]. Furthermore,

> most arrhythmias are harmless but if they are "re-entrant in nature and [...] occur because of the spatial distribution of cardiac tissue" they can seriously hamper the pumping mechanism of the heart and lead to death [17, p. 401].

As a result, a fuller understanding of spiral wave dynamics in these media becomes imperative.

The equivariant dynamical system approach

In recent years, one of the most rewarding approach to the study of spiral waves is based on Barkley's initial observation that the observed transition from rotating to modulated rotating wave can be explained *via* a Hopf bifurcation together with the underlying Euclidean symmetries of the governing reaction-diffusion equations [1, 2] (*i.e.*: the semi-flow generated by the dynamical system commutes with the

$$u(t,x) \longmapsto u(t, x_1 \cos\theta - x_2 \sin\theta + p_1, x_1 \sin\theta + x_2 \cos\theta + p_2), \tag{1.1}$$

where $(\theta, p_1, p_2) \in \mathbb{S}^1 \times \mathbb{R}^2 \simeq \mathbb{SE}(2)$ and $x \in \mathbb{R}^2$ [10,38]). This lead Barkley to formulate a simpler *ad hoc* 5-dimensional ODE system with Euclidean symmetry replicating the above transition [3].

Sandstede, Scheel and Wulff then proved a general center manifold reduction theorem (CMRT) for relative equilibria and relative periodic solutions in spatially extended infinite-dimensional Euclidean-equivariant dynamical systems, providing a mathematical justification of Barkley's insight [11,30–33]. However, this center manifold reduction

theorem requires that the spiral wave satisfy certain spectral gap conditions, which often fail [34]. Sandstede and Scheel have developed a comprehensive theory of spiral instabilities using techniques of spatial dynamics [28, 29] to deal with such a situation.

Other methods are also used to reduce the dynamics to finite-dimensional systems (such as the kinematic model using the curvature of the wave as a driving mechanism [24]), but the equivariant dynamical system approach has the advantage that it can often provide universal, model-independent explanations and predictions regarding the dynamics and bifurcations of spiral waves. For example, the fore-mentioned 'Hopf bifurcation' from rigid rotation to quasi-periodic meandering has been observed in numerically [4] and experimentally [21]. Another example is provided by the anchoring/repelling of spiral waves on/from a site of inhomogeneity, which has been observed in numerical integrations of an Oregonator system [25], in photo-sensitive chemical reactions [40] and in cardiac tissue [9]: using a model-independent approach based on forced symmetry-breaking, LeBlanc and Wulff showed that anchoring/repelling of rotating waves is a generic property of systems in which the translation symmetry is broken by a small perturbation [19].[1] In the same vein, certain dynamics of spiral waves in anisotropic media, such as phase-locking and linear drifting of meandering spiral, have been shown to be generic consequences of rotational symmetry-breaking [18, 26, 27].

The basic viewpoint

Consider a piece of cardiac tissue on which numerous (roughly) circular ablation have been performed, perhaps in order to treat a patient who is suffering from atrial fibrillation [14, 23]. These surgical procedures affects both the geometry and the excitability of the tissue. Under certain modeling assumptions, any system used to model the electrical activity of the tissue needs to incorporate translational symmetry-breaking (TSB) components to model the effects of the circular ablations, and a rotational symmetry-breaking (RSB) component to model the effects of anisotropy. Let us model the electrical properties of such a perturbed piece of anisotropic cardiac tissue using

[1] In this paper, we use the terms 'generic' and 'typical' interchangeably: the set of coefficient values for which the anchoring/repelling property fails to hold has measure zero in the complete coefficient space.

a modified version of the bidomain equations of cardiology, under the modeling assumption that the circular ablation (inhomogeneous) zones consist of a finite number of independent "sources" which are localized near distinct sites $\zeta_1, \ldots, \zeta_n$ in the plane (see [7] for a similar hypothesis). The model then has the form

$$\begin{aligned}
u_t &= \frac{1}{\varsigma}(u - \frac{u^3}{3} - v) + \nabla^2 u + \frac{\alpha\varepsilon}{1+\alpha(1-\varepsilon)}\Psi_{x_1x_1} + \sum_{j=1}^{n}\mu_j g_j^u(\|x-\zeta_j\|^2, \mu) \\
v_t &= \varsigma(u + \delta - \gamma v) + \sum_{j=1}^{n}\mu_j g_j^v(\|x-\zeta_j\|^2, \mu), \\
\nabla^2\Psi &+ \varepsilon g(\alpha,\varepsilon)\Psi_{x_2x_2} = \varepsilon h(\alpha,\varepsilon)u_{x_2x_2},
\end{aligned} \tag{1.2}$$

where u is a transmembrane potential, v controls the recovery of the action potential, Ψ is an auxiliary potential (without obvious physical interpretation), x_1 is the preferred direction in physical space in which tissue fibers align, ε is a measure of that preference, g and h are appropriate model functions, $\alpha, \varsigma, \delta$ and γ are model parameters, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ is a small parameter and $g_j^{u,v}$ are smooth functions, uniformly bounded in their variables [5, 8, 20, 26].

If the tissue has equal anisotropy ratios (*i.e.* $\varepsilon = 0$) and the inhomogeneities have no effect on spiral wave dynamics (*i.e.* $\mu = 0$), (1.2) decouples into the FitzHugh-Nagumo equations for u and v, and Poisson's equation for Ψ [26].

Let $\mathbb{SE}(2)$ denote the group of all planar translations and rotations, and fix an integer $1 \leq \jmath^*$ and $\zeta \in \mathbb{R}^2$. The subgroups $\mathbb{Z}_{\jmath^*}\dot{+}\mathbb{R}^2$ (the notation will be explained later) and $\mathbb{SO}(2)_\zeta$ of $\mathbb{SE}(2)$ consist of all cartesian pairs of translations and rotations about the origin by an integer multiple of $2\pi/\jmath^*$ radians, and of all rotations about the point ζ, respectively. Let $\Gamma = \mathbb{C}\dot{+}\mathbb{Z}_{\jmath^*}$ or $\Gamma = \mathbb{SO}(2)_\zeta$. Then, $\Gamma < \mathbb{SE}(2)$ and we will say that the semi-flow $\Psi_{t,\varepsilon,\mu}$ is $\Gamma-$equivariant if it commutes with the restriction of (1.1) to Γ.

In the equivariant dynamical system approach, the particular form of the functions $g_j^{u,v}$ is unimportant; the analysis is driven by the fact that (1.2) can sustain spiral wave propagation [5,8,20,26] and by the equivariance properties of the semi-flow $\Phi_{t,\varepsilon,\mu}$ generated by (1.2), namely: if we neglect boundary effects, the semi-flow

(E1) is $\mathbb{SE}(2)-$equivariant when $(\varepsilon, \mu) = 0$;

(E2) is $\mathbb{Z}_2 \dot{+} \mathbb{R}^2-$equivariant when $\varepsilon \neq 0$ is small and $\mu = 0$;

(E3) preserves rotations around ζ_i (but generically not translations) when $\varepsilon = 0$ and $\mu_j = 0$ for all $j \neq i$, and

(E4) is (generically) trivially equivariant when (ε, μ) is a generic small parameter vector.

This is but a special case of a more general family of semi-flows for which **(E2)** is replaced by the following property: the semi-flow

(E2') is $\mathbb{Z}_{\jmath^*} \dot{+} \mathbb{R}^2-$equivariant when $\varepsilon \neq 0$ is small and $\mu = 0$ for some integer $\jmath^* \geq 1$.

In [5–7], we used the dynamical system approach to study spiral anchoring in media satisfying **(E1)**, **(E2')**, **(E3)** and **(E4)**: the predictive power of the method was used to show that in the case $n > 1$, spiral anchoring typically takes place *away* from the inhomogeneities. At the time, such a statement defied experimental wisdom.

Epicyclic drifting

At this stage, nothing has been said about the nominal topic of this paper: epicyclic drifting. The various spiral motions observed in experiments and simulations have been classified according to their tip path, an arbitrary point on the wave front that is followed in time [9,21]: for instance, the tip path of a (rigidly) rotating wave is a perfect circle. Barkley [4] and Wulff [38] have shown that the appearance of an epicyclic tip path can be linked to a 'symmetric Hopf bifurcation:' when that happens, every spiral wave in the excitable medium is epicyclic.

However, other epicyclic behaviour cannot be explained by this mechanism. When the sizes of the physical domain and of the spiral core are comparable, the latter is sometimes attracted to the boundary of the domain and rotates around it in a meandering fashion. This has been observed in experiments and numerical simulations in a light-sensitive BZ reaction [39, 40].

Yet another instance of epicyclic motion is shown figure 1: in a bounded region, all solutions are attracted/repelled to/by an epicyclic solution manifold. This type of spiral wave motion is what we refer to as *epicyclic drifting.* À la Poincaré-Bendixson, if a system has a stable epicyclic manifolds (stable in the sense of Lyapunov) it will

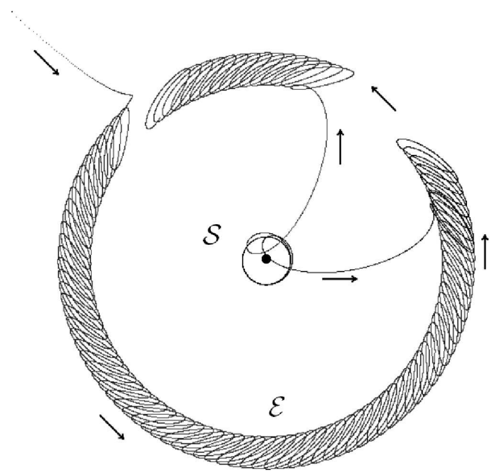

Figure 1: Epicyclic motion on the stable epicyclic manifold $\mathcal{E}$. The arrows indicate the direction of the flow, while $\mathcal{S}$ corresponds to a repelling (perturbed) rotating wave solution pinned at the inhomogeneity indicated by the black dot (see [7] for details on spiral anchoring).

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also have a repelling rotating wave (see figure 1), and *vice-versa*. As such, these manifolds cannot be observed in fully Euclidean media. What then, can forced Euclidean symmetry-breaking (FESB) tell us about epicyclic drifting in systems with the equivariance properties of **(E1)**, **(E2')**, **(E3)** and **(E4)**? The only work in this vein has been performed by LeBlanc and Wulff in [19], in the case $n = 1$ and without rotational symmetry-breaking: unfortunately, the main tool in their analysis cannot be used in the general case.

Article Overview

The main object of analysis in the present paper is a finite-dimensional system of ODE that share the equivariance properties of **(E1)**, **(E2')**, **(E3)** and **(E4)** when $n > 1$: it is derived in section 2.1. Then, in section 2.3, we present a preliminary result about averaging which will subsequently be used to prove our main results: to wit, when $\varepsilon = 0$ or $\jmath^* = 1$ and certain conditions are satisfied, there is a (minimal) parameter wedge region in which an epicyclic manifold persists. In the case $\jmath^* > 1$, the

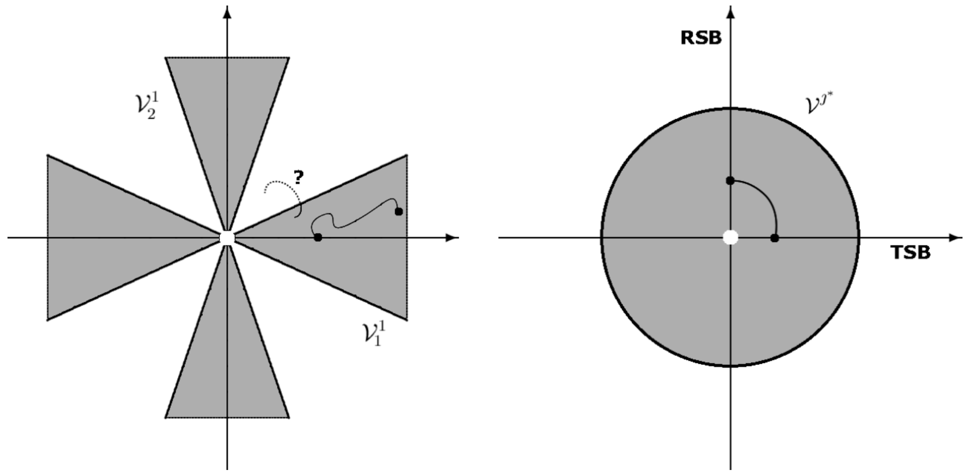

Figure 2: On the left, the epicyclic parameter wedge regions for the case $n = 2$ without anisotropy. On the right, the epicyclic deleted neighbourhood in parameter space for the case $n = 1$ with anisotropy characterized by $\jmath^* > 1$. If the semi-flow has an epicyclic solution manifold for a particular set of parameter values, then the semi-flow has an epicyclic solution manifold (of the same stability type) for all parameter values in the adjacent region. Note that these manifolds continuously deform along any path contained entirely in the parameter region. The local analysis does not provide a clear picture of the behaviour as a path leaves a parameter region. In [5], for instance, we give an example where the epicyclic solution manifold disappears as a result of a saddle-node bifurcation of rotating waves. The notation will be explained later in the paper.

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epicyclic manifold persists in a deleted neighbourhood of the origin. The parameter wedges are illustrated in figure 2, for the case $n = 2$. The conditions needed depend on the kind of forced symmetry-breaking under consideration: in section 3, we study the general semi-flow under n TSB terms (*i.e.* $n > 1$, $\varepsilon = 0$); in section 4, we study the combination of a single RSB and a single TSB term (*i.e.* $n = 1$, $\varepsilon \neq 0$). We then combine these results in section 5 to obtain the epicyclic drifting theorems under general FESB. Next, we perform a simple numerical experiment on 1.2) with $n = 1$ showing the predicted epicyclic motion: to the best of our knowledge, the figure in section 6 is the first observed instance of epicyclic drifting in a numerical simulation of excitable media. Finally, we give the proofs of two technical results in appendix A.

2 Preliminaries

We start with a derivation of the appropriate center bundle equations describing the essential dynamics of spiral waves near a rotating wave under full Euclidean symmetry-breaking (FESB). More details on these manipulations can be found in [5,7,30,31,38]. Then, in order to lighten the text, we introduce some necessary definitions. Finally, we state an averaging theorem which will be used in later sections of this work.

2.1 The Center Bundle Equations

In order to facilitate the subsequent analysis, we make the same simplifying assumptions and adopt the same notation as in [6,7].

In particular, let X be a Banach space, $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ a neighborhood of the origin, and $\Phi_{t,\varepsilon,\mu}$ be a smoothly parameterized family (parameterized by $(\varepsilon, \mu) \in \mathcal{U}$) of smooth local semi-flows on X, and let

$$a : \mathbb{SE}(2) \longrightarrow \mathrm{GL}(X) \tag{2.1}$$

be a faithful and isometric representation of $\mathbb{SE}(2)$ in the space of bounded, invertible linear operators on X. For example, if X is a space of functions with planar domain, a typical $\mathbb{SE}(2)$ action (such as (1.1) in the preceding section) is given by

$$(a(\gamma)u)(x) = u(\gamma^{-1}(x)), \quad \gamma \in \mathbb{SE}(2).$$

In this paper, we concern ourselves with the study of epicycle drifting in the case where the two following hypotheses are satisfied. The first one is simply a re-telling of the equivariance properties **(E1)**, **(E2')**, **(E3)** and **(E4)**, while the second postulates the existence of a rotating wave in the unperturbed $\mathbb{SE}(2)-$equivariant semi-flow.

Hypothesis 1 *There exists $1 \leq \jmath^* \in \mathbb{N}$, distinct points $\zeta_1, \ldots, \zeta_n$ in $\mathbb{R}^2$ such that if e_j denotes the j^{th} vector of the canonical basis in $\mathbb{R}^n$, then $\forall\, u \in X,\ \varepsilon \neq 0,\ \alpha \neq 0,\ t > 0$,*

$$\begin{aligned}
\Phi_{t,\varepsilon,0}(a(\gamma)u) = a(\gamma)\Phi_{t,\varepsilon,0}(u) &\iff \gamma \in \mathbb{Z}_{\jmath^*}\dot{+}\mathbb{R}^2, \\
\Phi_{t,0,\alpha e_j}(a(\gamma)u) = a(\gamma)\Phi_{t,0,\alpha e_j}(u) &\iff \gamma \in \mathbb{SO}(2)_{\zeta_j}, \quad \textit{and} \\
\Phi_{t,0,0}(a(\gamma)u) = a(\gamma)\Phi_{t,0,0}(u), &\quad \forall\, \gamma \in \mathbb{SE}(2).
\end{aligned}$$

Hypothesis 2 *There exists* $u^* \in X$ *(with trivial isotropy subgroup) and* Ω^* *in the Lie algebra of* $\mathbb{SE}(2)$ *such that* $e^{\Omega^* t}$ *is a rotation and* $\Phi_{t,0}(u^*) = a(e^{\Omega^* t})u^*$ *for all* t*. Moreover, the set* $\{\, \mu \in \mathbb{C} \mid |\mu| \geq 1 \,\}$ *is a spectral set for the linearization* $a(e^{-\Omega^*})D\Phi_{1,0}(u^*)$ *with projection* P_* *such that the generalized eigenspace* range(P_*) *is three dimensional.*

As discussed previously, such semi-flows can arise from the family of perturbed reaction-diffusion systems (1.4) from [7] if $\varepsilon = 0$, as well as from the modified bidomain model (1.2) given in the introduction if $\jmath^* = 2$, for instance.

It has been shown in [5–7] that, for small parameter vectors $(\varepsilon, \mu) \in \mathbb{R} \times \mathbb{R}^n$, the essential dynamics of the semi-flow $\Phi_{t,\varepsilon,\mu}$ near a (hyperbolic) rotating wave is (locally) equivalent to the semi-flow of the following ordinary differential equations on the bundle $\mathbb{C} \times \mathbb{S}^1$:

$$\dot{p} = e^{it}\left[v + \beta G(t,\beta) + \sum_{j=1}^{n} \lambda_j H_j((p-\xi_j)e^{-it}, \overline{(p-\xi_j)}e^{it}, \lambda)\right] \tag{2.2}$$

where $v \in \mathbb{C}$, $(\beta, \lambda) \in \mathbb{R} \times \mathbb{R}^n$, $\xi_1, \ldots, \xi_n \in \mathbb{C}$ are all distinct and the functions G, H_j are smooth, periodic in t and uniformly bounded in p, and G is $2\pi/\jmath^*-$periodic in t. The specific form of the perturbations is a consequence of forced Euclidean symmetry-breaking from the $\mathbb{SE}(2)-$equivariance of (2.2) under the following $\mathbb{SE}(2)-$action on the bundle $\mathbb{C} \times \mathbb{S}^1$:

$$(x,\theta) \cdot (p,\varphi) = (e^{i\theta}p + x, \varphi + \theta), \tag{2.3}$$

for all $(p,\varphi) \in \mathbb{C} \times \mathbb{S}^1$ and $(x,\theta) \in \mathbb{SE}(2) = \mathbb{C} \times \mathbb{S}^1$, where $\mathbb{SE}(2) = \mathbb{C} \dot{+} \mathbb{S}^1$ with multiplication $(p_1,\varphi_1)\cdot(p_2,\varphi_2) = (e^{i\varphi_1}p_2 + p_1, \varphi_1 + \varphi_2)$. The non-standard multiplication is made explicit by using the semi-direct product notation $\dot{+}$.

Let $\jmath^* \geq 1$ be an integer and $\xi \in \mathbb{C}$. In $\mathbb{SE}(2) = \mathbb{C} \dot{+} \mathbb{S}^1$, the subgroup of rotations around ξ is given by

$$\mathbb{S}^1_\xi = \{(\xi,0)\cdot(0,\theta)\cdot(-\xi,0) : \theta \in \mathbb{S}^1\},$$

while the subgroup containing all translations and rotations by angle $\frac{2\pi k}{\jmath^*}$, $k \in \mathbb{Z}$, is

$$\mathbb{C} \dot{+} \mathbb{Z}_{\jmath^*} = \left\{ \left(x, \frac{2\pi k}{\jmath^*} \right) : k \in \mathbb{Z} : x \in \mathbb{C} \right\}.$$

Then $\mathbb{C}\dot{+}\mathbb{Z}_{\jmath^*} \simeq \mathbb{Z}_{\jmath^*}\dot{+}\mathbb{R}^2$ and $\mathbb{S}^1_\xi \simeq \mathbb{SO}(2)_\zeta$: under the action described by (2.3), the center bundle equation (2.2) is

(C1) $\mathbb{C}\dot{+}\mathbb{S}^1-$equivariant when $(\beta,\lambda)=0$;

(C2') $\mathbb{C}\dot{+}\mathbb{Z}_{\jmath^*}-$equivariant when $\beta \neq 0$ is small and $\lambda = 0$;

(C3) $\mathbb{S}^1_{\xi_\ell}-$equivariant when $\beta = 0$ and $\lambda_j = 0$ for all $j \neq \ell$, and

(C4) (generically) trivially equivariant when (β,λ) is a generic small parameter vector.

Clearly, (2.2) shares the equivariance properties of Hypothesis 1. As such, G 'models' the RSB perturbation while the various H_j 'model' the various TSB terms.

It might seem strange that the parameters (ε,μ) are replaced by (β,λ) in (2.2), just as the $\zeta_j \in \mathbb{R}^2$ are replaced by $\xi_j \in \mathbb{C}$, but since the center manifold reduction theorems of [30–33] do not provide an explicit relation between the coefficients of the original system of partial differential equations and the reduced ordinary differential system of center bundle equations, one cannot conclude that the parameters are the same in both systems.

2.2 Definitions

An integral manifold is *stable* if it has a neighbourhood in which all originating positive-time solutions approach the manifold exponentially; it is *hyperbolic* if the linearization of the flow on this manifold admits no critical eigenvalues. Let $\alpha_0 > 0$, $\Delta > 0$, $V \subseteq \mathbb{R}^p$, $\Sigma = \mathbb{R} \times V \times [0,\alpha_0]$, $f : \Sigma \to \mathbb{R}^q$, $g : V \to \mathbb{R}^q$ and $h : \mathbb{R} \times V \to \mathbb{R}^q$. We say that f is *Lipschitz in Hale's sense*, which we denote by $f \in \mathrm{Lip}(x;\Sigma,\eta(\alpha,V))$, if f is continuous in all of its arguments and is Lipschitz in x for $(t,x,\alpha) \in \Sigma$ with continuous Lipschitz constant.

Next we say that g is *bounded by* Δ *over* V, which we denote by $g \in \mathcal{B}(\Delta;V)$, if $\|g(x)\| \leq \Delta$ for all $x \in V$. Finally, we denote the fact that h is $T-$periodic in $\phi \in \mathbb{R}$ by $h \in \mathfrak{P}^T_\phi$. When the sets Σ and V are understood from the context, they are omitted. Finally, by abuse of notation, we shall often denote $O\left(|x_1| + \cdots + |x_m|\right)$ by $O(x_1,\ldots,x_m)$.

2.3 A Generalized Averaging Theorem

Averaging methods are used to determine whether a particular system has an non-trivial invariant integral manifold by studying an averaged system. The main theorem is a modified version of one of Hale's averaging theorems (see [5] for details); it can easily be extended to the case where ν is a parameter vector in $\mathbb{R}^n$.

Theorem 2.1 (modified from [15], theorem 6.1, pp. $526-527$) *Let* $\sigma_0 > 0$. *Consider the system of equations*

$$\begin{aligned}\dot{x} &= \epsilon\gamma_{\epsilon,\nu}x + \epsilon\Lambda(t,\psi,x,\epsilon,\nu)\\ \dot{\psi} &= d(\epsilon,\nu) + \Theta(t,\psi,x,\epsilon,\nu),\end{aligned} \tag{2.4}$$

where $\psi \in \mathbb{R}$, $x \in [-\sigma_0, \infty)$, $\gamma_{\epsilon,\nu} \neq 0$ *depends continuously on* (ϵ,ν), *and* d *is defined over* $S_0 = [-\epsilon_0, \epsilon_0] \times [-\nu_0, \nu_0]$, *with* $d(0,0) = 1$. *For* $\sigma > 0$, *let*

$$\Sigma_\sigma = \mathbb{R} \times \mathbb{R} \times [-\sigma, \sigma] \times S_0 \quad \text{and} \quad \Sigma_0 = \mathbb{R} \times \mathbb{R} \times \{0\} \times S_0.$$

Suppose $\Theta, \Lambda \in \mathfrak{P}_t^\chi \cap \mathfrak{P}_\psi^\omega$ *and that*

(i) Θ *and* Λ *are real-valued over* Σ_{σ_0};

(ii) $\Theta, \Lambda \in \mathcal{B}(\Xi(\epsilon,\nu);\Sigma_0)$ *where* $\Xi(\epsilon,\nu) = O(\epsilon,\nu)$;

(iii) for all $0 \leq \sigma \leq \sigma_0$, $\Theta \in \mathrm{Lip}(\psi, x; \Sigma_\sigma, \theta(\epsilon,\nu,\sigma))$ *and* $\Lambda \in \mathrm{Lip}(\psi, x; \Sigma_\sigma, \eta(\epsilon,\nu,\sigma))$, *with* $\theta(\epsilon,\nu,\sigma) = O(\epsilon,\nu,\sigma)$ *and* $\eta(\epsilon,\nu,\sigma) = O(\epsilon,\nu,\sigma)$.

Then, there exists $(\epsilon_1,\nu_1) \in (0,\epsilon_0] \times (0,\nu_0]$ *such that for all*

$$(\epsilon,\nu) \in S_1 = [-\epsilon_1, \epsilon_1] \times [-\nu_1, \nu_1]$$

with $\epsilon \neq 0$, *(2.4) has a hyperbolic integral manifold* $\mathcal{T}_{\epsilon,\nu}$ *which can be represented as an invariant torus* $x = \Upsilon_{\epsilon,\nu}(t,\psi)$, *where*

$$\Upsilon_{\epsilon,\nu} \in \mathcal{B}(D(\epsilon,\nu)) \cap \mathrm{Lip}(\psi, \Omega(\epsilon,\nu)) \cap \mathfrak{P}_t^\chi \cap \mathfrak{P}_\psi^\omega,$$

with $D(\epsilon,\nu), \Omega(\epsilon,\nu) \to 0$ *uniformly as* $(\epsilon,\nu) \to 0$. *Furthermore, the stability of* $\mathcal{T}_{\epsilon,\nu}$ *is exactly determined by the sign of* $\epsilon\gamma_{0,0}$.

3 Epicyclic Drifting For n Simultaneous TSB Terms

When $\beta = 0$, (2.2) gives the dynamics near a hyperbolic rotating wave for a parameterized family of semi-flows $\Phi_{t,0,\lambda}$ satisfying the forced-symmetry breaking conditions in hypothesis 1. We start with a brief review of epicyclic drifting in the case $n = 1$, which was studied in detail in [19], and then present our new results in the general case $n > 1$.

3.1 The Case $n = 1$

Without loss of generality, we may assume $\xi_1 = 0$. In this case, the center bundle equation (2.2) reduce to

$$\dot{p} = e^{it}\left[v + \lambda_1 H_1(pe^{-it}, \overline{p}e^{it}, \beta)\right], \tag{3.1}$$

where $v \in \mathbb{C}^\times$ and $\lambda_1 \in \mathbb{R}$ is small. Set $\widetilde{H}(w, \overline{w}, \lambda_1) = H_1(w - iv, \overline{w} + i\overline{v}, \lambda_1)$.

Theorem 3.1 ([19], re − written to fit the current symbolism) *Let*

$$I(\rho) = \mathrm{Re}\left[\int_0^{2\pi} e^{-it}\widetilde{H}\left(\rho e^{-it}, \rho e^{it}, 0\right) dt\right].$$

If $\rho_0 > 0$ is a hyperbolic solution of $I(\rho) = 0$, then for all $\lambda_1 \neq 0$ small enough, the center bundle equation (3.1) has an integral (solution) manifold $\mathcal{E}^1_{\lambda_1}$ around the origin, whose stability is exactly determined by the sign of $\lambda_1 I'(\rho_0)$.

These solutions represent quasi-periodic motion around the origin in the $p-$plane and are observable as epicycle-like motion along a circular boundary in the physical space, with angular frequency $1 + O(\lambda_1)$. Note that the hypotheses of theorem 3.1 are not generic: in a random system, $I(\rho)$ may very well not have a positive hyperbolic root.

The presence of a repelling integral manifold could explain the fact that spirals are sometimes observed to be repulsed by an inhomogeneity if the spiral tip is located beyond a certain distance from the perturbation center [25].

3.2 The Case $n > 1$

However, the main averaging tool used in [19] to obtained theorem (3.1) cannot be used to analyze the situation in the case $n > 1$; furthermore, this difficulty yields a interesting twist, as we shall see in this section.

By re-labeling the indices in (2.2) if necessary, we can temporarily shift our point of view so that ξ_1 plays the central role in the following analysis. Set $\Xi_j = \xi_j - \xi_1$ for $j = 1, \ldots, n$. Then, under the co-rotating frame of reference $z = p - \xi_1 + ie^{it}v$, (2.2) becomes

$$\dot{z} = e^{it} \sum_{j=1}^{n} \lambda_j H_j\big((z - \Xi_j)e^{-it} - iv, \overline{(z - \Xi_j)}e^{it} + i\overline{v}, \lambda\big). \tag{3.2}$$

When $\lambda_1 \neq 0$ and $\lambda_2 = \cdots = \lambda_n = 0$, we find ourselves in the situation described in the previous subsection. Now, set $\epsilon = \lambda_1$, $\nu_1 = 1$ and $\lambda_j = \nu_j \epsilon$ for $j = 2, \ldots, n$, and $\nu = (\nu_2, \ldots, \nu_n) \in \mathbb{R}^{n-1}$. Then (3.2) can be viewed as a perturbation of the corresponding equation in the case $n = 1$. Note that $\Xi_1 = 0$ and $\lambda = (1, \nu)\epsilon$.

Equation (3.2) rewrites as

$$\dot{z} = \epsilon e^{it} \sum_{j=1}^{n} \nu_j H_j\big((z - \Xi_j)e^{-it} - iv, \overline{(z - \Xi_j)}e^{it} + i\overline{v}, (1, \nu)\epsilon\big). \tag{3.3}$$

Let $\hat{H}_j(w, \overline{w}, \epsilon, \nu) = H_j\big(w - iv, \overline{w} + i\overline{v}, (1, \nu)\epsilon\big)$ for $j = 1, \ldots n$. Then (3.3) becomes

$$\dot{z} = \epsilon e^{it} K(ze^{-it}, \overline{z}e^{it}, t, \epsilon, \nu) \tag{3.4}$$

where $K(w, \overline{w}, t, \epsilon, \nu) = \sum_{j=1}^{n} \nu_j \hat{H}_j(w - \Xi_j e^{-it}, \overline{w} - \overline{\Xi}_j e^{it}, \epsilon, \nu)$ is $2\pi-$periodic in t. Consider the near-identity change of variables

$$z = w + \epsilon \kappa(w, \overline{w}, t, \epsilon, \nu) \tag{3.5}$$

where $\kappa \in \mathfrak{P}_t^{2\pi}$ is differentiable in all of its variables. Then

$$\dot{z} = \dot{w} + \epsilon \left(\frac{\partial \kappa}{\partial t} + \frac{\partial \kappa}{\partial w}\dot{w} + \frac{\partial \kappa}{\partial \overline{w}}\dot{\overline{w}} \right).$$

Introducing the equivalent complex conjugate equation, this last system becomes

$$\left[I_2 + \epsilon \begin{pmatrix} \kappa_w & \kappa_{\overline{w}} \\ \overline{\kappa}_w & \overline{\kappa}_{\overline{w}} \end{pmatrix}\right] \begin{pmatrix} \dot{w} \\ \dot{\overline{w}} \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \dot{\overline{z}} \end{pmatrix} - \epsilon \begin{pmatrix} \kappa_t \\ \overline{\kappa}_t \end{pmatrix}, \tag{3.6}$$

where κ_w, $\kappa_{\overline{w}}$, κ_t, $\overline{\kappa}_w$, $\overline{\kappa}_{\overline{w}}$, $\overline{\kappa}_t$ are used to denote the partial derivatives of κ and $\overline{\kappa}$. Set

$$\mathcal{I} = I_2 + \epsilon \begin{pmatrix} \kappa_w & \kappa_{\overline{w}} \\ \overline{\kappa}_w & \overline{\kappa}_{\overline{w}} \end{pmatrix}.$$

Combining (3.6) with (3.4) yields

$$\begin{pmatrix} \dot{w} \\ \dot{\overline{w}} \end{pmatrix} = \epsilon \mathcal{I}^{-1} \begin{pmatrix} e^{it} K\left((w+\epsilon\kappa)e^{-it}, (\overline{w}+\epsilon\overline{\kappa})e^{it}, t, \epsilon, \nu\right) - \kappa_t \\ e^{-it}\overline{K}\left((w+\epsilon\kappa)e^{-it}, (\overline{w}+\epsilon\overline{\kappa})e^{it}, t, \epsilon, \nu\right) - \overline{\kappa}_t \end{pmatrix} \tag{3.7}$$

By Taylor's theorem, there are appropriate continuous bounded functions A_1, A_2 and $A_3 \in \mathfrak{P}_t^{2\pi}$ satisfying

$$\begin{aligned} e^{it} K\left((w+\epsilon\kappa)e^{-it}, (\overline{w}+\epsilon\overline{\kappa})e^{it}, t, \epsilon, \nu\right) &= e^{it} K\left(we^{-it}, \overline{w}e^{it}, t, 0, \nu\right) + \epsilon A_1(w, \overline{w}, t, \epsilon, \nu) \\ \kappa_t(w, \overline{w}, t, \epsilon, \nu) &= \kappa_t(w, \overline{w}, t, 0, \nu) + \epsilon A_2(w, \overline{w}, t, \epsilon, \nu) \end{aligned}$$

and

$$\mathcal{I}^{-1} = \begin{pmatrix} 1 - \epsilon\kappa_w^0 & -\epsilon\kappa_{\overline{w}}^0 \\ -\epsilon\overline{\kappa}_w^0 & 1 - \epsilon\overline{\kappa}_{\overline{w}}^0 \end{pmatrix} + \epsilon^2 A_3(w, \overline{w}, t, \epsilon, \nu),$$

where

$$\kappa_w^0 = \kappa_w(w, \overline{w}, t, 0, \nu) \quad \text{and} \quad \kappa_{\overline{w}}^0 = \kappa_{\overline{w}}(w, \overline{w}, t, 0, \nu).$$

With these, (3.7) re-writes (upon dropping the equivalent complex conjugate equation) as

$$\dot{w} = \epsilon\left(e^{it} K(we^{-it}, \overline{w}e^{it}, t, 0, \nu) - \kappa_t(w, \overline{w}, t, 0, \nu)\right) + \epsilon^2 \mathcal{H}(w, \overline{w}, t, \epsilon, \nu), \tag{3.8}$$

where $\mathcal{H} \in \mathfrak{P}_t^{2\pi}$ is bounded and continuous in all its variables. Denote the average value of $e^{it}K(we^{-it}, \overline{w}e^{it}, t, 0, \nu)$ by

$$M^1(w, \overline{w}, \nu) = \frac{1}{2\pi} \int_0^{2\pi} e^{it} K(we^{-it}, \overline{w}e^{it}, t, 0, \nu)\, dt. \tag{3.9}$$

Then

$$e^{it}K(we^{-it},\overline{w}e^{it},t,0,\nu) = M^1(w,\overline{w},\nu) + F(w,\overline{w},t,\nu),$$

where $F \in \mathfrak{P}_t^{2\pi}$ is uniformly continuous and

$$\int_0^{2\pi} F(w,\overline{w},t,\nu)\,dt = 0. \tag{3.10}$$

Let κ be an antiderivative of F with respect to t. Then $\kappa \in \mathfrak{P}_t^{2\pi}$ by (3.10) and

$$F(w,\overline{w},t,\nu) - \kappa_t(w,\overline{w},t,0,\nu) = 0.$$

With such a κ, (3.8) simplifies to

$$\dot{w} = \epsilon M^1(w,\overline{w},\nu) + \epsilon^2 \mathcal{H}(w,\overline{w},t,\epsilon,\nu). \tag{3.11}$$

It is easy to see that $M^1(w,\overline{w},0)$ is $\mathbb{S}^1-$equivariant (see appendix for details); as such, there is a continuous function $L_1 : \mathbb{R} \to \mathbb{C}$ such that $M^1(w,\overline{w},0) = wL_1(w\overline{w})$ [12, p. 360].

By Taylor's theorem, there are appropriate continuous bounded functions M_j, for $j = 2,\ldots,n$, such that

$$M^1(w,\overline{w},\nu) = M^1(w,\overline{w},0) + \sum_{j=2}^{n} \nu_j M_j(w,\overline{w},\nu)$$

and so (3.11) becomes

$$\dot{w} = \epsilon w L_1(w\overline{w}) + \epsilon W(w,\overline{w},t,\epsilon,\nu), \tag{3.12}$$

where

$$W(w,\overline{w},t,\epsilon,\nu) = \sum_{j=2}^{n} \nu_j M_j(w,\overline{w},\nu) + \epsilon\mathcal{H}(w,\overline{w},t,\epsilon,\nu). \tag{3.13}$$

Differentiating the polar coordinates $w = \rho e^{-i(\psi - t)}$ yields

$$\begin{aligned}\dot{\rho} &= \operatorname{Re}\left[\dot{w}e^{i(\psi-t)}\right]\\ \dot{\psi} &= 1 - \frac{1}{\rho}\operatorname{Im}\left[\dot{w}e^{i(\psi-t)}\right].\end{aligned}$$

But

$$\begin{aligned}
\dot{w}e^{i(\psi-t)} &= (\epsilon w L_1(w\overline{w}) + \epsilon W(w,\overline{w},t,\epsilon,\nu))\, e^{i(\psi-t)} \\
&= \left(\epsilon\rho e^{-i(\psi-t)} L_1(\rho^2) + \epsilon W(\rho e^{-i(\psi-t)}, \rho e^{i(\psi-t)}, t,\epsilon,\nu)\right) e^{i(\psi-t)} \\
&= \epsilon\rho L_1(\rho^2) + \epsilon e^{i(\psi-t)} W(\rho e^{-i(\psi-t)}, \rho e^{i(\psi-t)}, t,\epsilon,\nu)
\end{aligned}$$

and so

$$\begin{aligned}
\dot{\rho} &= \epsilon R_0^1(\rho) + \epsilon R(t,\psi,\rho,\epsilon,\nu) \\
\dot{\psi} &= 1 + \epsilon\Psi_0(\rho) + \epsilon\Psi(t,\psi,\rho,\epsilon,\nu),
\end{aligned} \tag{3.14}$$

where $R_0^1(\rho) = \rho\,\mathrm{Re}\,[L_1(\rho^2)]$, $\Psi_0(\rho) = -\,\mathrm{Im}\,[L_1(\rho^2)]$ and

$$\begin{aligned}
R(t,\psi,\rho,\epsilon,\nu) &= \mathrm{Re}\left[e^{i(\psi-t)} W(\rho e^{-i(\psi-t)}, \rho e^{i(\psi-t)}, t,\epsilon,\nu)\right] \\
\Psi(t,\psi,\rho,\epsilon,\nu) &= -\frac{1}{\rho}\,\mathrm{Im}\left[e^{i(\psi-t)} W(\rho e^{-i(\psi-t)}, \rho e^{i(\psi-t)}, t,\epsilon,\nu)\right].
\end{aligned} \tag{3.15}$$

Note that $R, \Psi \in \mathfrak{P}_t^{2\pi} \cap \mathfrak{P}_\psi^{2\pi}$ and that Ψ is not defined at $\rho = 0$. We now give sufficient conditions for the existence of an integral manifold in (3.14).

Theorem 3.2 *Assume that R and Ψ, as defined in (3.15), are C^1 on intervals away from $\rho = 0$ and that the averaged equation*

$$\dot{\rho} = \epsilon R_0^1(\rho) \tag{3.16}$$

has an equilibrium $\rho_1 > 0$ with $D_\rho R_0^1(\rho_1) = \gamma_1 \neq 0$. If the parameters are small enough to satisfy the conditions outlined in the proof below, then (3.14) has an invariant torus $\hat{\mathcal{E}}_{\epsilon,\nu}$, whose stability is exactly determined by the sign of $\epsilon\gamma_1$.

Proof: By the implicit function theorem, there is a neighbourhood

$$U = (-\epsilon_*, \epsilon_*) \times \prod_{j=2}^{n} (-\nu_{j,*}, \nu_{j,*})$$

in parameter space and a continuous function $\rho : U \to \mathbb{R}^+$ such that $\rho(0,0) = \rho_1$,

$$\epsilon R_0^1(\rho(\epsilon,\nu)) \equiv 0 \quad \text{and} \quad D_\rho R_0^1(\rho(\epsilon,\nu)) = \gamma_{\epsilon,\nu} \neq 0,$$

where $\gamma_{\epsilon,\nu}\gamma_1 > 0$ for all $(\epsilon, \nu) \in U$, *i.e.* the stability of the equilibria $\rho(\epsilon, \nu)$ is the same as that of ρ_0 for all $(\epsilon, \nu) \in U$.

When $\epsilon = 0$, the phase space of (3.14) is foliated by invariant tori and so, from now on, we will assume that $\epsilon \neq 0$. Consider the change of variables $\rho = \rho(\epsilon, \nu) + x$ in (3.14). Differentiating the new coordinates, we get $\dot{x} = \dot{\rho}$ and the equivalent system

$$\begin{aligned} \dot{x} &= \epsilon R_0^1(\rho(\epsilon,\nu) + x) + \epsilon R(t, \psi, \rho(\epsilon, \nu) + x, \epsilon, \nu) \\ \dot{\psi} &= 1 + \epsilon \Psi_0(\rho(\epsilon, \nu) + x) + \epsilon \Psi(t, \psi, \rho(\epsilon, \nu) + x, \epsilon, \nu). \end{aligned}$$

By Taylor's theorem, there are continuously differentiable functions B_1 and B_2 such that

$$\begin{aligned} R_0^1(\rho(\epsilon,\nu) + x) &= R_0^1(\rho(\epsilon,\nu)) + D_x R_0^1(\rho(\epsilon,\nu))x + B_1(x,\epsilon,\nu)x^2 \\ \Psi_0(\rho(\epsilon,\nu) + x) &= \Psi_0(\rho(\epsilon,\nu)) + B_2(x,\epsilon,\nu)x. \end{aligned}$$

Since $R_0^1(\rho(\epsilon,\nu)) \equiv 0$ and $D_x R_0^1(\rho(\epsilon,\nu)) = \gamma_{\epsilon,\nu}$, we obtain the new system

$$\begin{aligned} \dot{x} &= \epsilon \gamma_{\epsilon,\nu} x + \epsilon \Lambda(t, \psi, x, \epsilon, \nu) \\ \dot{\psi} &= d(\epsilon, \nu) + \Theta(t, \psi, x, \epsilon, \nu), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \Lambda(t,\psi,x,\epsilon,\nu) &= B_1(x,\epsilon,\nu)x^2 + R(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) \\ \Theta(t,\psi,x,\epsilon,\nu) &= \epsilon B_2(x,\epsilon,\nu)x + \epsilon\Psi(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) \\ d(\epsilon,\nu) &= 1 + \epsilon \Psi_0(\rho(\epsilon,\nu)) \end{aligned}$$

are at least C^1 by hypothesis.

Let $U^+ = \{\varsigma \in U : \varsigma_i > 0 \text{ for all } i = 1, \dots, n\}$ and $(\epsilon_0, \nu_0) \in U^+$. Define

$$S_0 = [-\epsilon_0, \epsilon_0] \times \prod_{j=2}^{n} [-\nu_{0,j}, \nu_{0,j}].$$

As $\rho_1 > 0$ and $\rho(\epsilon, \nu)$ is continuous on U, it is possible to chose (ϵ_0, ν_0) in such a way that

$$\sigma_0 = \min_{(\epsilon,\nu)\in S_0} \{\rho(\epsilon,\nu)\} - \tfrac{1}{2}\rho_1 > 0.$$

If $x \geq -\sigma_0$, then $\rho = \rho(\epsilon, \nu) + x \geq \rho(\epsilon, \nu) - \sigma_0 \geq \frac{1}{2}\rho_1$ for all $(\epsilon, \nu) \in S_0$. In that case, Θ and Λ are continuously differentiable, as R and Ψ are continuously differentiable in ρ on $[\frac{1}{2}\rho_1, \infty)$. Note further that $\Theta, \Lambda \in \mathfrak{P}_t^{2\pi} \cap \mathfrak{P}_\psi^{2\pi}$.

Set $\Sigma_0 = \mathbb{R} \times \mathbb{R} \times \{0\} \times S_0$, and $\Sigma_\sigma = \mathbb{R} \times \mathbb{R} \times [-\sigma, \sigma] \times S_0$. Then

1. Θ and Λ are bounded by a function $\Xi(\epsilon, \nu) = O(\epsilon, \nu_2, \ldots, \nu_n)$ over Σ_0 (see appendix for details), and
2. for all $0 \leq \sigma \leq \sigma_0$, Θ and Λ are Lipschitz in Hale's sense (with Lipschitz constants $\theta(\epsilon, \nu, \delta) = O(\epsilon, \nu_2, \ldots, \nu_n, \delta)$ and $\eta(\epsilon, \nu, \delta) = O(\epsilon, \nu_2, \ldots, \nu_n, \delta)$, respectively) over Σ_σ (see appendix for details).

Accordingly, theorem 2.1 can be applied to show there is a neighbourhood $S_1 \subseteq S_0$ of the origin in parameter space for which (3.14) (since it is equivalent to (3.17)) has an invariant torus $\hat{\mathcal{T}}_{\epsilon,\nu}$ when $(\epsilon, \nu) \in S_1$. Furthermore, the stability of $\hat{\mathcal{T}}_{\epsilon,\nu}$ is the same as that of the hyperbolic equilibrium $\rho(\epsilon, \nu)$, which is given by $\epsilon\gamma_1$. $\square$

The invariant torus $\hat{\mathcal{T}}_{\epsilon,\nu}$ appearing in the proof of Theorem 3.2 can be parameterized by a relation of the form $x = \Upsilon_{\epsilon,\nu}(\theta_1, \theta_2)$, where $\theta_1, \theta_2 \in \mathbb{S}^1$. Let

$$\langle \hat{\mathcal{T}}_{\epsilon,\nu} \rangle = \frac{1}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} \Upsilon_{\epsilon,\nu}(\theta_1, \theta_2)\, d\theta_1 d\theta_2 \tag{3.18}$$

denote the *center* of $\hat{\mathcal{T}}_{\epsilon,\nu}$, and let $\hat{\mathcal{E}}_{\epsilon,\nu}$ be the corresponding *epicyclic manifold* of (3.4), in which all solutions are epicycles when projected upon the $z-$plane.

Define the average value

$$[\hat{\mathcal{E}}_{\epsilon,\nu}]_{\mathrm{D}} = \frac{1}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} \Big(\big(\rho(\epsilon, \nu) + \langle \hat{\mathcal{T}}_{\epsilon,\nu} \rangle\big) e^{-i(\psi - t)} \\ + \epsilon\kappa\Big(\big(\rho(\epsilon, \nu) + \langle \hat{\mathcal{T}}_{\epsilon,\nu} \rangle\big) e^{-i(\psi - t)}, \text{c.c.}, t, \epsilon, \nu\Big)\Big)\, d\psi dt. \tag{3.19}$$

If $\hat{\mathcal{T}}_{\epsilon,\nu}$ is stable (in the sense of theorem 2.1), we shall say that $[\hat{\mathcal{E}}_{\epsilon,\nu}]_{\mathrm{D}}$ is the *center of drifting* of $\hat{\mathcal{E}}_{\epsilon,\nu}$.

Theorem 3.3 *Suppose the hypotheses of theorem 3.2 are satisfied. Then there exists a wedge-shaped region near* $\lambda = 0$ *of the form*

$$\mathcal{V}_1 = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \,:\, |\lambda_j| < V_{1,j}|\lambda_1|,\ \ V_{1,j} > 0,\ \text{for } j \neq 1 \text{ and } \lambda_1 \text{ near } 0\}$$

such that for all $0 \neq \lambda \in \mathcal{V}_1$, *(2.2) has an epicycle manifold* $\mathcal{E}_\lambda^1$, *with* $[\mathcal{E}_\lambda^1]_\mathrm{D}$ *near, but generically not at,* ξ_1. *Furthermore,* $[\mathcal{E}_\lambda^1]_\mathrm{D}$ *is a center of drifting when* $\lambda_1\gamma_1 < 0$.

Proof: According to theorem 3.2, there are constants $\epsilon_1, \nu_{1,2}, \dots, \nu_{1,n} > 0$ and a neighbourhood

$$S_1 = [-\epsilon_1, \epsilon_1] \times \prod_{j=2}^{n} [-\nu_{1,j}, \nu_{1,j}]$$

such that (3.14) has an integral manifold $\hat{\mathcal{E}}_{\epsilon,\nu}$ whenever $(\epsilon, \nu) \in S_1$.
For $j \neq 1$, set $\lambda_1 = \epsilon \neq 0$, $\lambda_j = \nu_j \epsilon$ and $V_{1,j} = \nu_{1,j}$. Then $\lambda \in \mathcal{V}_1$ as

$$|\lambda_j| \leq |\nu_j| \cdot |\lambda_1| \leq V_{1,j}|\lambda_1| \quad \text{for } j \neq 1,$$

and (2.2) has an integral manifold $\mathcal{E}_\lambda^1 = \xi_1 - ie^{it}v + \hat{\mathcal{E}}_{\epsilon,\nu}$. Furthermore, $[\mathcal{E}_\lambda^1]_\mathrm{D} = \xi_1 + O(\lambda_1)$ and so $[\mathcal{E}_\lambda^1]_\mathrm{D} \neq \xi_1$ for a generic $0 \neq \lambda \in \mathcal{V}_1$. The conclusion on the stability of $\mathcal{E}_\lambda^1$ then follows directly from theorem 3.2. □

Remark 3.4

1. These isolated epicycle manifolds need not in general be unique for a given $\lambda \in \mathcal{V}_1$ as $R_0^1(\rho) = 0$ may have any number of hyperbolic solutions.
2. In generic semi-flows, all that can be said with certainty from the analysis when the parameter values stray outside of $\mathcal{V}_1$ is that the epicycle manifolds in (2.2) drift away from ξ_1, which cannot then be a center of drifting. This is not unlike the situation with regards to spiral anchoring [7]. Richer dynamics and interactions with rotating waves can also take place; for instance in [5], we gave an example in which the epicyclic manifold collapses at a saddle-node bifurcation of rotating waves.
3. Note that the actual parameter region in which epicyclic drifting is observed may be much larger than $\mathcal{V}_1$: however, our local analysis cannot be used to obtain global results.

The preceding results have been achieved by considering (2.2) under a co-rotating frame of reference around ξ_1. Of course, since the choice for ξ_1 was arbitrary, corresponding results must also be achieved, in exactly the same manner, when the viewpoint shifts

to another ξ_k. Indeed, for $j = 1, \ldots, n$, define the average functions

$$M^j(w, \overline{w}) = \frac{1}{2\pi} \int_0^{2\pi} e^{it} \hat{H}_j(we^{-it}, \overline{w}e^{it}, 0, 0)\, dt;$$

as before, each M^j is $\mathbb{S}^1-$equivariant and so there are continuous functions $L_j : \mathbb{R} \to \mathbb{C}$ such that $M^j(w, \overline{w}) = wL_j(w\overline{w})$. We will call

$$R_0^j(\rho) = \rho \operatorname{Re}\left[L_j(\rho^2)\right]$$

the *epicycle functions of* (2.2).

Corollary 3.5 *Let $k \in \{1, \ldots, n\}$. If $\rho_* > 0$ is such that*

$$R_0^k(\rho_*) = 0 \quad \textit{and} \quad D_\rho R_0^k(\rho_*) = \gamma_* \neq 0,$$

then there exists a wedge-shaped region near $\lambda = 0$ of the form

$$\mathcal{V}_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \,:\, |\lambda_j| < V_{k,j}|\lambda_k|,\;\; V_{k,j} > 0,\; \text{for } j \neq k \text{ and } \lambda_k \text{ near } 0\,\}$$

such that for all $0 \neq \lambda \in \mathcal{V}_k$, (2.2) has an epicycle manifold $\mathcal{E}_\lambda^k$, with $[\mathcal{E}_\lambda^k]_{\mathrm{D}}$ near, but generically not at, ξ_k. Furthermore, $[\mathcal{E}_\lambda^k]_{\mathrm{D}}$ is a center of drifting when $\lambda_k\gamma_* < 0$.

Proof: The epicycle function R_0^k is exactly the function that would appear in (3.16) had the preceding analysis been done around ξ_k. Theorems 3.2 and 3.3 can then be applied directly to obtain the desired result. □

Clearly, the remarks appearing after the proof of theorem 3.3 still hold. There is one last statement to be made concerning epicycle manifolds: theorem 3.5 only gives sufficient conditions for their existence in (2.2). In section [5], we have provided an example that shows that they are not, in fact, necessary conditions.

4 Epicyclic Drifting For Combined RSB-TSB Terms

In this section, we investigate another way in which the Euclidean symmetry can be broken: by combining rotational and translational symmetry breaking. In effect, we are lifting the restriction $\beta = 0$, with $n = 1$ in (2.2).

It turns out that the value of $\jmath^*$ plays a crucial role in the analysis: the cases $\jmath^* = 1$ and $\jmath^* > 1$ are essentially different. The general lines are very similar to those of the preceding section, as such, the proofs are omitted in order to avoid tedious repetitions. In either case, however, me assume without loss of generality that $\xi_1 = 0$.

4.1 The Case $\jmath^* = 1$

Let $F_G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ be defined by

$$F_G(t, \beta) = e^{it}\Big[-iv + \beta \sum_{m \neq -1} \frac{g_m(\beta)e^{imt}}{i(m+1)} \Big],$$

where the $g_m(\beta)$ are the Fourier coefficients of $G \in \mathfrak{P}_t^{2\pi}$. Set $z = p - F_G$. Then, (2.2) rewrites as

$$\dot{z} = \beta g_{-1}(\beta) + \beta e^{it} H((z + F_G(t, \beta)e^{-it}, \text{c.c.}, \lambda_1), \tag{4.1}$$

where c.c. represents throughout the complex conjugate of the preceding term. Generically, $g_{-1}(0) \neq 0$. Set $\epsilon = \beta$, $\nu = \lambda_1$ and $\epsilon = \hat{\epsilon}\lambda_1$. Then (4.1) transforms to

$$\dot{z} = \nu H_*(ze^{-it}, \overline{z}e^{it}, t, \hat{\epsilon}, \nu), \tag{4.2}$$

where

$$H_*(w, \overline{w}, t, \hat{\epsilon}, \nu) = \hat{\epsilon} g_{-1}(\hat{\epsilon}\nu) + e^{it} H(w + F_G(t, \hat{\epsilon}\nu)e^{-it}, \text{c. c.}, \nu) \tag{4.3}$$

is $2\pi-$periodic in t, smooth and uniformly bounded in w. Consider the near-identity change of variables

$$z = w + \nu\varrho(w, \overline{w}, t, \epsilon, \nu) \tag{4.4}$$

where $\varrho \in \mathfrak{P}_t^{2\pi}$ is continuous in all of its variables and to be determined later. This change of variables transform (4.2) into the equivalent system

$$\dot{w} = \nu \left(H_*(we^{-it}, \overline{w}e^{it}, t, \hat{\epsilon}, 0) - \varrho_t(w, \overline{w}, t, \hat{\epsilon}, 0) \right) + \nu^2 \mathcal{H}_*(w, \overline{w}, t, \hat{\epsilon}, \nu), \tag{4.5}$$

where $\mathcal{H}_* \in \mathfrak{P}_t^{2\pi}$ is bounded and continuous in all its variables. Denote the average value of $H_*(we^{-it}, \overline{w}e^{it}, t, \hat{\epsilon}, 0)$ by

$$M_*(w, \overline{w}, \hat{\epsilon}) = \frac{1}{2\pi} \int_0^{2\pi} H_*(we^{-it}, \overline{w}e^{it}, t, \hat{\epsilon}, 0)\, dt.$$

Since

$$\begin{aligned} H_*(we^{-it}, \overline{w}e^{it}, t, \hat{\epsilon}, 0) &= \hat{\epsilon} g_{-1}(0) + e^{it} H\left((w + F_G(t, 0))e^{-it}, \text{c. c.}, 0\right) \\ &= \hat{\epsilon} g_{-1}(0) + e^{it} H(we^{-it} - iv, \overline{w}e^{it} + i\overline{v}, 0), \end{aligned}$$

we have

$$M_*(w, \overline{w}, \hat{\epsilon}) = \hat{\epsilon} g_{-1}(0) + \frac{1}{2\pi} \int_0^{2\pi} e^{it} H(we^{-it} - iv, \overline{w}e^{it} + i\overline{v}, 0)\, dt.$$

Then

$$H_*(we^{-it}, \overline{w}e^{it}, t, \hat{\epsilon}, 0) = M_*(w, \overline{w}, \hat{\epsilon}) + F_*(w, \overline{w}, t, \hat{\epsilon}),$$

where $F_* \in \mathfrak{P}_t^{2\pi}$ is uniformly continuous and

$$\int_0^{2\pi} F(w, \overline{w}, t, \hat{\epsilon})\, dt = 0. \tag{4.6}$$

Let ϱ be an antiderivative of F_* with respect to t. Then $\varrho \in \mathfrak{P}_t^{2\pi}$ by (4.6) and

$$F_*(w, \overline{w}, t, \hat{\epsilon}) - \varrho_t(w, \overline{w}, t, \hat{\epsilon}, 0) = 0.$$

With such a ϱ, (4.5) simplifies to

$$\dot{w} = \nu M_*(w, \overline{w}, \hat{\epsilon}) + \nu^2 \mathcal{H}_*(w, \overline{w}, t, \hat{\epsilon}, \nu). \tag{4.7}$$

As $M_*(w, \overline{w}, 0)$ is also $\mathbb{S}^1-$equivariant, there is a continuous function $L_* : \mathbb{R} \to \mathbb{C}$ such that $M_*(w, \overline{w}, 0) = wL_*(w\overline{w})$, and so (4.7) becomes

$$\dot{w} = \nu w L_*(w\overline{w}) + \nu W_*(w, \overline{w}, t, \hat{\epsilon}, \nu), \tag{4.8}$$

where

$$W_*(w, \overline{w}, t, \hat{\epsilon}, \nu) = \hat{\epsilon} g_{-1}(0) + \nu \mathcal{H}_*(w, \overline{w}, t, \hat{\epsilon}, \nu). \tag{4.9}$$

Differentiating the polar coordinates $w = \rho e^{-i(\psi - t)}$ and substituting in 4.8 yields

$$\begin{aligned} \dot{\rho} &= \nu R_*(\rho) + \nu R(t, \psi, \rho, \hat{\epsilon}, \nu) \\ \dot{\psi} &= 1 + \nu \Psi_*(\rho) + \nu \Psi(t, \psi, \rho, \hat{\epsilon}, \nu), \end{aligned} \tag{4.10}$$

where $R_*(\rho) = \rho \operatorname{Re}[L_*(\rho^2)]$, $\Psi_*(\rho) = -\operatorname{Im}[L_*(\rho^2)]$, $R, \Psi \in \mathfrak{P}_t^{2\pi} \cap \mathfrak{P}_\psi^{2\pi}$ and Ψ is not continuous at $\rho = 0$. We now provide sufficient conditions for the existence of an epicycle manifold in (2.2).

Theorem 4.1 *Assume that R and Ψ, as defined in (4.10), are C^1 on intervals away from $\rho = 0$ and that the averaged equation*

$$\dot{\rho} = \epsilon R_*(\rho) \tag{4.11}$$

has an equilibrium $\rho_0 > 0$ with $D_\rho R_*(\rho_0) = \gamma_0 \neq 0$. If the parameters are small enough, there exists a wedge-shaped region near $(\beta, \lambda_1) = (0, 0)$ of the form

$$\mathcal{V} = \{(\beta, \lambda_1) \in \mathbb{R}^2 \,:\, |\beta| < K|\lambda_1|,\ \ K > 0,\ \ \lambda_1 \textit{ near } 0\,\}$$

such that for all $(\beta, \lambda_1) \in \mathcal{V}$, $\beta \neq 0$, (2.2) has an epicycle manifold $\mathcal{G}^1_{\beta,\lambda_1}$, with $[\mathcal{G}^1_{\beta,\lambda_1}]_{\mathrm{D}}$ near, but generically not at, the origin. Furthermore, $[\mathcal{G}^1_{\beta,\lambda_1}]_{\mathrm{D}}$ is a center of drifting when $\lambda_1 \gamma_0 < 0$.

The remarks after theorem 3.3 still hold after having been suitably modified.

4.2 The Case $\jmath^* > 1$

The case $\jmath^* > 1$ is handled slightly differently. Let $C^0_{\mathbb{R}}(\mathbb{C})$ and $C^1_{\mathbb{R}}(\mathbb{C})$ be the spaces of continuous and continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$, respectively. Then

$$C^0_{2\pi/\jmath^*} = \{f : f \in \mathfrak{P}_t^{2\pi/\jmath^*} \cap C^0_{\mathbb{R}}(\mathbb{C})\} \quad \text{and} \quad C^1_{2\pi/\jmath^*} = \{f : f \in \mathfrak{P}_t^{2\pi/\jmath^*} \cap C^1_{\mathbb{R}}(\mathbb{C})\}$$

are Banach spaces when endowed with the respective norms

$$||u||_0 = \sup\{|u(x)| : x \in [0, 2\pi/\jmath^*]\} \quad \text{and} \quad ||u||_1 = ||u||_0 + ||u'||_0,$$

and the linear operator $\mathcal{Y} : C^1_{2\pi/\jmath^*} \to C^0_{2\pi/\jmath^*}$ defined by $\mathcal{Y}(u) = iu + u'$ is bounded, invertible and has bounded inverse.

Define the nonlinear operator $\mathcal{H}_G : C^1_{2\pi/\jmath^*} \times \mathbb{R}^2 \to C^0_{2\pi/\jmath^*}$ by

$$\mathcal{H}_G(u, \beta, \lambda_1) = \mathcal{Y}(u) - \lambda_1 H \left(u - iv + \beta \sum_{m\in\mathbb{Z}} \frac{g_m(\beta) e^{im\jmath^* t}}{i(m\jmath^* + 1)}, \text{c.c.}, \lambda_1 \right), \tag{4.12}$$

where the $g_m(\beta)$ are as they were in the previous section. But $\mathcal{H}_G(0,0,0) = 0$ and $D_1\mathcal{H}_G(0,0,0) = i \neq 0$ and so, by the implicit function theorem, there is a neighbourhood $\mathcal{N}$ of the origin in $\mathbb{R}^2$ and a unique smooth function $U : \mathbb{R}^2 \to C^1_{2\pi/\jmath^*}$ satisfying $U(0,0) = 0$ and $\mathcal{H}_G(U(\beta, \lambda_1), \beta, \lambda_1) \equiv 0$ for all $(\beta, \lambda_1) \in \mathcal{N}$.

Let $F_G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ be defined by

$$F_G(t, \beta, \lambda_1) = e^{it} \Big[- iv + \beta \sum_{m\in\mathbb{Z}} \frac{g_m(\beta) e^{im\jmath^* t}}{i(m\jmath^* + 1)} + U(\beta, \lambda_1)(t) \Big]. \tag{4.13}$$

Then $\mathcal{Y}(U(\beta, \lambda_1))(t) = \lambda_1 H\left(F_G(t, \beta, \lambda_1)e^{-it}, \text{c.c.}, \lambda_1\right)$, and, upon setting $z = p - F_G$, (2.2) rewrites as

$$\dot{z} = \lambda_1 e^{it} \big[H((z + F_G(t, \beta, \lambda_1))e^{-it}, \text{c.c.}, \lambda_1) - H(F_G(t, \beta, \lambda_1)e^{-it}, \text{c.c.}, \lambda_1)) \big],$$

which reduces to

$$\dot{z} = \lambda_1 e^{it} \widehat{H}(ze^{-it}, \overline{z}e^{it}, t, \beta, \lambda_1), \tag{4.14}$$

where

$$\widehat{H}(w, \overline{w}, t, \beta, \lambda_1) = H(w + F_G(t, \beta, \lambda_1)e^{-it}, \text{c.c.}, \lambda_1) - H(F_G(t, \beta, \lambda_1)e^{-it}, \text{c.c.}, \lambda_1) \tag{4.15}$$

is $2\pi/\jmath^*-$periodic in t. Then, (4.14) becomes

$$\begin{aligned} \dot{\rho} &= \lambda_1 R_*(\rho) + \lambda_1 R(t, \psi, \rho, \beta, \lambda_1) \\ \dot{\psi} &= 1 + \lambda_1 \Psi_*(\rho) + \lambda_1 \Psi(t, \psi, \rho, \beta, \lambda_1), \end{aligned} \tag{4.16}$$

where $R_*(\rho) = \rho \operatorname{Re}[L_*(\rho^2)]$, $\Psi_*(\rho) = -\operatorname{Im}[L_*(\rho^2)]$ for some continuous function $L_* : \mathbb{R} \to \mathbb{C}$, $R, \Psi \in \mathfrak{P}^{2\pi}_t \cap \mathfrak{P}^{2\pi}_\psi$ and Ψ is not continuous at $\rho = 0$.

Theorem 4.2 *Assume that R and Ψ, as defined in (4.16), are C^1 on intervals away from $\rho = 0$ and that the averaged equation*

$$\dot{\rho} = \epsilon R_*(\rho) \tag{4.17}$$

has an equilibrium $\rho_0 > 0$ with $D_\rho R_*(\rho_0) = \gamma_0 \neq 0$. If the parameters are in a (small enough) deleted neighbourhood $\mathcal{V}^{\jmath^*}$ of the origin, (2.2) has an epicyclic manifold $\mathcal{G}^{\jmath^*}_{\beta,\lambda_1}$, with $[\mathcal{G}^{\jmath^*}_{\beta,\lambda_1}]_{\mathrm{D}} = 0$. Furthermore, the origin is a center of drifting when $\lambda_1\gamma_0 < 0$.

Remark 4.3

1. The small term $\hat{\epsilon}g_{-1}(0)$ in (4.3) and the absence of a corresponding term in (4.15) are responsible for the different form of the regions $\mathcal{V}$ (in theorem 4.11) and $\mathcal{V}^{\jmath^*}$ (in theorem 4.17), as well as for the location of the center of drifting.
2. The remarks made after theorem 3.3 still hold, when suitably modified.
3. There are a lot of similarities between our analysis and the results obtained during the analysis of spiral anchoring in [6,7], such as the presence of wedge-shaped regions or the deleted neighbourhoods, depending on the nature of $\jmath^*$. In particular, one might hope that the epicyclic manifolds would possess $\mathbb{Z}_{\jmath^*}-$spatio-temporal symmetry; however, this is not the case as the averaged system defined by (4.16) is generally only $2\pi/\jmath^*-$periodic in t when $R \equiv 0$ and $\Psi \equiv 0$. That being said, the epicycles themselves possess this symmetry in an appropriate co-rotating frame of reference.

5 Epicyclic Drifting For General ESB Terms

Lifting all restrictions on β and λ in (2.2), and combining the methods of the preceding section, we obtain the following general epicyclic drifting theorems for (2.2).

Theorem 5.1 *Let $n > 1$ and $k \in \{1, \dots, n\}$. Given a hyperbolic equilibrium ρ_* of an appropriate averaged equation $\dot{\rho} = \beta\tilde{R}^k_0(\rho)$ with eigenvalue γ^*_k (derived as in section 3), there is a region in parameter space near $(\beta = \lambda_0, \lambda_1, \dots, \lambda_n) = 0$ of the form*

$$\mathcal{V}^1_k = \{(\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1} : |\lambda_j| < V_{j,k}|\lambda_k|,\ V_{j,k} > 0,\ \text{for } j \neq k \text{ and } \lambda_k \text{ near } 0\}$$

when $\jmath^* = 1$, or of the form

$$\mathcal{V}_k^{\jmath^*} = \{(\beta, \lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n+1} \,:\, |\beta| < \beta_0,\ |\lambda_j| < V_{j,k}|\lambda_k|,\ V_{j,k} > 0,\ \text{for } j \neq k \text{ and } \lambda_k \text{ near } 0\,\}$$

for some constant $\beta_0 > 0$, when $\jmath^* > 1$, such that for all $0 \neq (\beta, \lambda) \in \mathcal{V}_k^{\jmath^*}$, with the additional condition that $\beta \neq 0$ when $\jmath^* = 1$, (2.2) has an epicyclic manifold $\mathcal{E}_{\beta,\lambda}^{\jmath^*;k}$, with $[\mathcal{E}_{\beta,\lambda}^{\jmath^*;k}]_{\mathrm{D}}$ near, but generically not at, ξ_k. Furthermore, $[\mathcal{E}_{\beta,\lambda}^{\jmath^*;k}]_{\mathrm{D}}$ is a center of drifting when $\lambda_k \gamma_k^* < 0$.

Since the hypotheses of this theorems are not generic, it is not clear that such integral epicyclic manifolds are common, and their existence must sometimes be inferred in the physical space, especially if they are repelling, such as appears to be the case in [25].

6 An Epicyclic Manifold in Physical Space

In this section, we provide what we believe to be the first observed example of epicyclic drifting in a modified bidomain system. Our system is a TSB perturbation of the bidomain equations and parameter values found in [27]:

$$\begin{aligned}
u_t &= \frac{1}{\varsigma}\left(u - \frac{u^3}{3} - v\right) + \Delta u + \frac{\alpha\varepsilon}{1+\alpha(1-\varepsilon)}\psi_{xx} \\
u_{yy} &= \left[\left(2+\alpha+\frac{1}{\alpha}\right)\psi_{xx} + \left(2+\alpha(1-\varepsilon)+\frac{1}{\alpha(1-\varepsilon)}\right)\psi_{yy}\right]\left(1+\frac{1}{\alpha(1-\varepsilon)}\right)^{-1}\frac{1}{\varepsilon} \\
v_t &= \varsigma(u+\delta-\gamma v) + \phi(x-35, y-35)
\end{aligned} \tag{6.1}$$

where $\varsigma = 0.3$, $\alpha = 1.0$, $\varepsilon = 0.75$, $\delta = 0.8$, $\gamma = 0.5$ and

$$\phi(z_1, z_2) = -0.03\exp\left(-0.085(z_1^2 + z_2^2)\right).$$

The TSB term $\phi(x-35, y-35)$ is uniformly bounded and goes to 0 as $\|(x,y)\| \to \infty$. Futhermore, it preserves rotations around the point $(35, 35)$.

As our emphasis lies with qualitative observations rather than with precise numerical analysis, the numerical perspective is somewhat naive. The computations are carried out on a two-dimensional square domain $[10, 60]^2$ with 120 grid points to a side

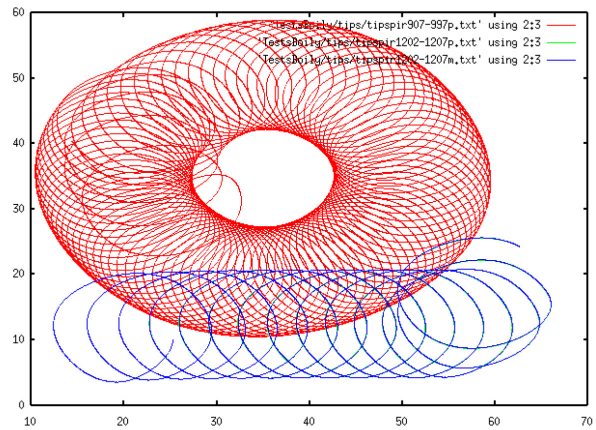

Figure 3: Tip path of the u component of two solutions of (6.1). The solution in red is attracted by an epicyclic manifold. The solution in blue shows the effect of the boundary: the two solutions are clearly not of the same nature. Compare the epicyclic manifold with the image in figure 1.

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and Neumann boundary condition, using a 5-point Laplacian, continuous linear finite elements on square meshes and the fully implicit second order Gear finite difference integrator. The tip path of the u component of two solutions are shown in figure 3.

7 Summary and Concluding Remarks

Recently, equivariant dynamical systems theory has been used to provide an approach to the study of spiral wave dynamics and bifurcations, in particular, it has provided mechanisms for such behaviour as spiral tip meandering and resonant growth [2,30–33, 38], spiral anchoring/repelling and boundary drifting [19], which have been explained as consequences of forced Euclidean symmetry breaking.

In this paper, we have used this model-independent approach to analyze epicyclic drifting of the spiral tip in media with several localized inhomogeneities, with or without anisotropy. The result of a simple numerical experiment is in agreement with our theoretical conclusions. The RSB terms are characterized by an integer $\jmath^* > 1$. It is important to note that, as of now, only the integers $\jmath^* = 2$ (anisotropic cardiac tissue,

say) and $\jmath^* = 1$ (an excitable medium that in which there is a directed current, such as a reaction-diffusion-advection system) have easy interpretations in the context of RSB.

It should be recalled that our analysis rests on certain simplifying modeling assumptions which may not be valid in some realistic physical systems: namely concerning the discrete nature of the inhomogeneities (i.e. finite number of inhomogeneity sites) and the hypothesis of local rotational symmetry of the individual inhomogeneities. In a realistic model of excitable media such as the bidomain model describing electrical conduction of cardiac tissue, with or without advection, actual inhomogeneities may lack this local rotational symmetry, or may even be distributed smoothly and non-symmetrically over the medium (*i.e.* an inhomogeneity field).

Should the discrete localized inhomogeneities not possess circular symmetry, we believe that our results would still describe the essential qualitative features of epicyclic drifting, even though our analysis does not technically apply, as epicyclic drifting is linked to hyperbolic fixed points. However, for a smoothly distributed non-symmetric inhomogeneity field, our techniques are unlikely to yield meaningful results.

Finally, we would like to point out that our existence results do not provide a description of the flow on the epicyclic manifolds: we gave two examples of center bundle systems with essentially different flows in [5]. In the first example, the flow is "ergodic" (see the first figure of this article): if the spiral tip lies in the epicyclic manifold, the tip path eventually fills the entire manifold. In the second example, the flow on the epicyclic manifold is dictated by the stability of rotating waves located *on* the manifold (see figure 4).

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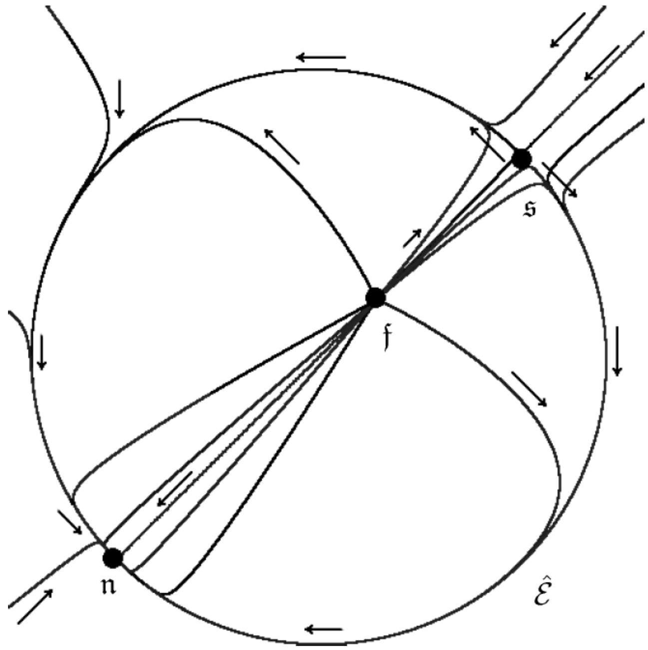

Figure 4: Projection (in the $z-$plane) of a stable epicycle manifold $\hat{\mathcal{E}}$ containing three rotating waves $\mathfrak{s}$, $\mathfrak{n}$ and $\mathfrak{f}$. The arrows indicate the flow on the manifold.

A Technical Results

Proposition A.1 *The function $M^1(w, \overline{w}, 0)$ defined in (3.9) is $\mathbb{S}^1-$equivariant.*

Proof: Recall that $\hat{H}_1$ is $\mathbb{S}^1-$equivariant by construction. Then

$$\begin{aligned} M^1(we^{-i\theta}, \overline{w}e^{i\theta}, 0) &= \frac{1}{2\pi}\int_0^{2\pi} e^{it} K(we^{-i\theta}e^{-it}, \overline{w}e^{i\theta}e^{it}, t, 0, 0)\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{it} \hat{H}_1(we^{-it}e^{-i\theta}, \overline{w}e^{it}e^{i\theta}, 0, 0)\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{it} e^{i\theta} \hat{H}_1(we^{-it}, \overline{w}e^{it}, 0, 0)\, dt \\ &= e^{i\theta} M^1(w, \overline{w}, 0), \end{aligned}$$

that is, M^1 is $\mathbb{S}^1-$equivariant. $\square$

Proposition A.2 *Let all terms, variables and functions be as in Theorem 3.2. In particular, the functions R and Ψ are C^1 on intervals away from $\rho = 0$. Denote*

$$\Sigma_0^r = [0, 2\pi] \times [0, 2\pi] \times \{0\} \times S_0$$
$$\Sigma_\sigma^r = [0, 2\pi] \times [0, 2\pi] \times [-\sigma, \sigma] \times S_0.$$

Note that these spaces, as well as the spaces Σ_0 and Σ_σ from Theorem 2.1, are convex. The functions Θ and Λ satisfy the following conditions:

1. *Θ and Λ are bounded by a function $\Xi(\epsilon, \nu) = O(\epsilon, \nu_2, \ldots, \nu_n)$ over Σ_0, and*
2. *for all $0 \le \sigma \le \sigma_0$, Θ and Λ are Lipschitz in Hale's sense (with Lipschitz constants $\theta(\epsilon, \nu, \sigma) = O(\epsilon, \nu_2, \ldots, \nu_n, \sigma)$ and $\nu(\epsilon, \nu, \sigma) = O(\epsilon, \nu_2, \ldots, \nu_n, \sigma)$, respectively) over Σ_σ.* (p. 18)

Proof: Since $\Theta, \Lambda \in \mathfrak{P}_t^{2\pi} \cap \mathfrak{P}_\psi^{2\pi}$, we need only show that the first statement holds over Σ_0^r and the second over Σ_σ^r, for all $0 \le \sigma \le \sigma_0$. For $j = 1, \ldots, n$, there are appropriate functions $R_j, \Psi_j \in \mathfrak{P}_t^{2\pi} \cap \mathfrak{P}_\psi^{2\pi}$, C^1 on intervals away from $\rho = 0$, such that

$$\begin{aligned} \Psi(t, \psi, \rho, \epsilon, \nu) &= \epsilon \Psi_1(t, \psi, \rho, \epsilon, \nu) + \sum_{j=2}^{n} \nu_j \Psi_j(t, \psi, \rho, \nu) \\ R(t, \psi, \rho, \epsilon, \nu) &= \epsilon R_1(t, \psi, \rho, \epsilon, \nu) + \sum_{j=2}^{n} \nu_j R_j(t, \psi, \rho, \nu), \end{aligned} \tag{A.1}$$

according to (3.13).

1. Over Σ_0^r, we have $x = 0$ and so

$$\begin{aligned} \Theta(t, \psi, 0, \epsilon, \nu) &= \epsilon \Psi(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) \\ &= \epsilon^2 \Psi_1(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) + \epsilon \sum_{j=2}^{n} \nu_j \Psi_j(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} \Lambda(t, \psi, 0, \epsilon, \nu) &= R(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) \\ &= \epsilon R_1(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) + \sum_{j=2}^{n} \nu_j R_j(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu) \end{aligned} \tag{A.3}$$

according to (A.1).

For $j = 1, \ldots, n$, the continuous functions $|R_j|$ and $|\epsilon\Psi_j|$ reach their maximum C_j and E_j, respectively, on the compact set $[0, 2\pi] \times \{0\}$ for $j = 1, \ldots, n$. Then

$$|\epsilon\Psi_j(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu)| \leq E_j \quad \text{and} \quad |R_j(t, \psi, \rho(\epsilon, \nu), \epsilon, \nu)| \leq C_j$$

over Σ_0^r for $j = 1, \ldots, n$. According to (A.2) and (A.3),

$$\begin{aligned} |\Theta(t, \psi, 0, \epsilon, \nu)| &\leq |\epsilon| E_1 + \sum_{j=2}^{n} |\nu_j| E_j = Q_1(\epsilon, \nu) \\ |\Lambda(t, \psi, 0, \epsilon, \nu)| &\leq |\epsilon| C_1 + \sum_{j=2}^{n} |\nu_j| C_j = Q_2(\epsilon, \nu) \end{aligned}$$

over Σ_0^r. Set

$$\Xi(\epsilon, \nu) = \max\{Q_1(\epsilon, \nu), Q_2(\epsilon, \nu)\}.$$

Then Λ and Θ are bounded by $\Xi(\epsilon, \nu) = O(\epsilon, \nu_2, \ldots, \nu_n)$ over Σ_0^r.

2. Let $(\psi_1, x_1), (\psi_2, x_2) \in \mathbb{R} \times [-\sigma, \sigma]$ for $0 \leq \sigma \leq \sigma_0$. By one of the mean value theorems, there exist points $(\psi^*, x^*), (\psi_*, x_*) \in [0, 2\pi] \times [-\sigma, \sigma]$ on the line joining (ψ_1, x_1) and (ψ_2, x_2) such that

$$\begin{aligned} |\Theta(t, \psi_1, x_1, \epsilon, \nu) - \Theta(t, \psi_2, x_2, \epsilon, \nu)| &= |\widehat{\Theta}(t, \psi^*, x^*, \epsilon, \nu)| \big[|\psi_1 - \psi_2| + |x_1 - x_2| \big] \\ |\Lambda(t, \psi_1, x_1, \epsilon, \nu) - \Lambda(t, \psi_2, x_2, \epsilon, \nu)| &= |\widehat{\Lambda}(t, \psi_*, x_*, \epsilon, \nu)| \big[|\psi_1 - \psi_2| + |x_1 - x_2| \big], \end{aligned}$$

where

$$\begin{aligned} \widehat{\Theta}(t, \psi, x, \epsilon, \nu) &= D_x\Theta(t, \psi, x, \epsilon, \nu) + D_\psi\Theta(t, \psi, x, \epsilon, \nu) \\ &= xK_0^\Psi(x, \epsilon, \nu) + \epsilon K_1^\Psi(t, \psi, x, \epsilon, \nu) + \sum_{j=2}^{n} \nu_j K_j^\Psi(t, \psi, x, \epsilon, \nu) \\ \widehat{\Lambda}(t, \psi, x, \epsilon, \nu) &= D_x\Lambda(t, \psi, x, \epsilon, \nu) + D_\psi\Lambda(t, \psi, x, \epsilon, \nu) \\ &= xK_0^R(x, \epsilon, \nu) + \epsilon K_1^R(t, \psi, x, \epsilon, \nu) + \sum_{j=2}^{n} \nu_j K_j^R(t, \psi, x, \epsilon, \nu), \end{aligned}$$

where

$$\begin{aligned}
K_0^{\Psi}(x,\epsilon,\nu) &= \epsilon D_x B_2(x,\epsilon,\nu)\\
K_0^{R}(x,\epsilon,\nu) &= \epsilon\left(D_x B_1(x,\epsilon,\nu)x + 2B_1(x,\epsilon,\nu)\right)\\
K_1^{\Psi}(t,\psi,x,\epsilon,\nu) &= B_2(x,\epsilon,\nu) + \epsilon\left(D_x\Psi_1(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) + D_\psi\Psi_1(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu)\right)\\
K_1^{R}(t,\psi,x,\epsilon,\nu) &= \epsilon\left(D_x R_1(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) + D_\psi R_1(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu)\right)
\end{aligned}$$

and

$$\begin{aligned}
K_j^{\Psi}(t,\psi,x,\epsilon,\nu) &= \epsilon\left(D_x\Psi_j(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) + D_\psi\Psi_j(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu)\right)\\
K_j^{R}(t,\psi,x,\epsilon,\nu) &= \epsilon\left(D_x R_j(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu) + D_\psi R_j(t,\psi,\rho(\epsilon,\nu)+x,\epsilon,\nu)\right),
\end{aligned}$$

where Ψ_j and R_j are as in (A.1). Since Θ and Λ are continuously differentiable, $\widehat{\Theta}$ and $\widehat{\Lambda}$ are continuous on Σ_σ^r, as are K_j^{Ψ} and K_j^R for $j = 0, \ldots, n$.

In particular, the functions $|K_j^{\Psi}|$ and $|K_j^{\Psi}|$ each reach their respective maximum k_j^{Ψ} and k_j^R on Σ_σ^r for $j = 0, \ldots, n$. Then, note that $|x^*|, |x_*| \le \sigma$,

$$\begin{aligned}
|\widehat{\Theta}(t,\psi^*,x^*,\epsilon,\nu)| &\le |x^*||K_0^{\Psi}(x^*,\epsilon,\nu)| + |\epsilon||K_1^{\Psi}(t,\psi^*,x^*,\epsilon,\nu)| + \sum_{j=2}^{n}|\nu_j||K_j^{\Psi}(t,\psi^*,x^*,\epsilon,\nu)|\\
&\le |x^*|k_0^{\Psi} + |\epsilon|k_1^{\Psi} + \sum_{j=2}^{n}|\nu_j|k_j^{\Psi}\\
&\le \sigma k_0^{\Psi} + |\epsilon|k_1^{\Psi} + \sum_{j=2}^{n}|\nu_j|k_j^{\Psi} = \theta(\epsilon,\nu,\sigma)
\end{aligned}$$

and

$$\begin{aligned}
|\widehat{\Lambda}(t,\psi_*,x_*,\epsilon,\nu)| &\le |x_*||K_0^{R}(x_*,\epsilon,\nu)| + |\epsilon||K_1^{R}(t,\psi_*,x_*,\epsilon,\nu)| + \sum_{j=2}^{n}|\nu_j||K_j^{R}(t,\psi_*,x_*,\epsilon,\nu)|\\
&\le |x_*|k_0^{R} + |\epsilon|k_1^{R} + \sum_{j=2}^{n}|\nu_j|k_j^{R}\\
&\le \sigma k_0^{R} + |\epsilon|k_1^{R} + \sum_{j=2}^{n}|\nu_j|k_j^{R} = \nu(\epsilon,\nu,\sigma).
\end{aligned}$$

In particular

$$\begin{aligned}
|\Theta(t,\psi_1,x_1,\epsilon,\nu) - \Theta(t,\psi_2,x_2,\epsilon,\nu)| &\le \theta(\epsilon,\nu_2,\ldots,\nu_n,\sigma)\big[|\psi_1-\psi_2| + |x_1-x_2|\big]\\
|\Lambda(t,\psi_1,x_1,\epsilon,\nu) - \Lambda(t,\psi_2,x_2,\epsilon,\nu)| &\le \nu(\epsilon,\nu_2,\ldots,\nu_n,\sigma)\big[|\psi_1-\psi_2| + |x_1-x_2|\big],
\end{aligned}$$

where $\theta(\epsilon, \nu, \sigma), \nu(\epsilon, \nu, \sigma) = O(\epsilon, \nu_2, \dots, \nu_n, \sigma)$. Hence Θ and Λ are Lipschitz in Hale's sense.

This completes the proof. $\square$

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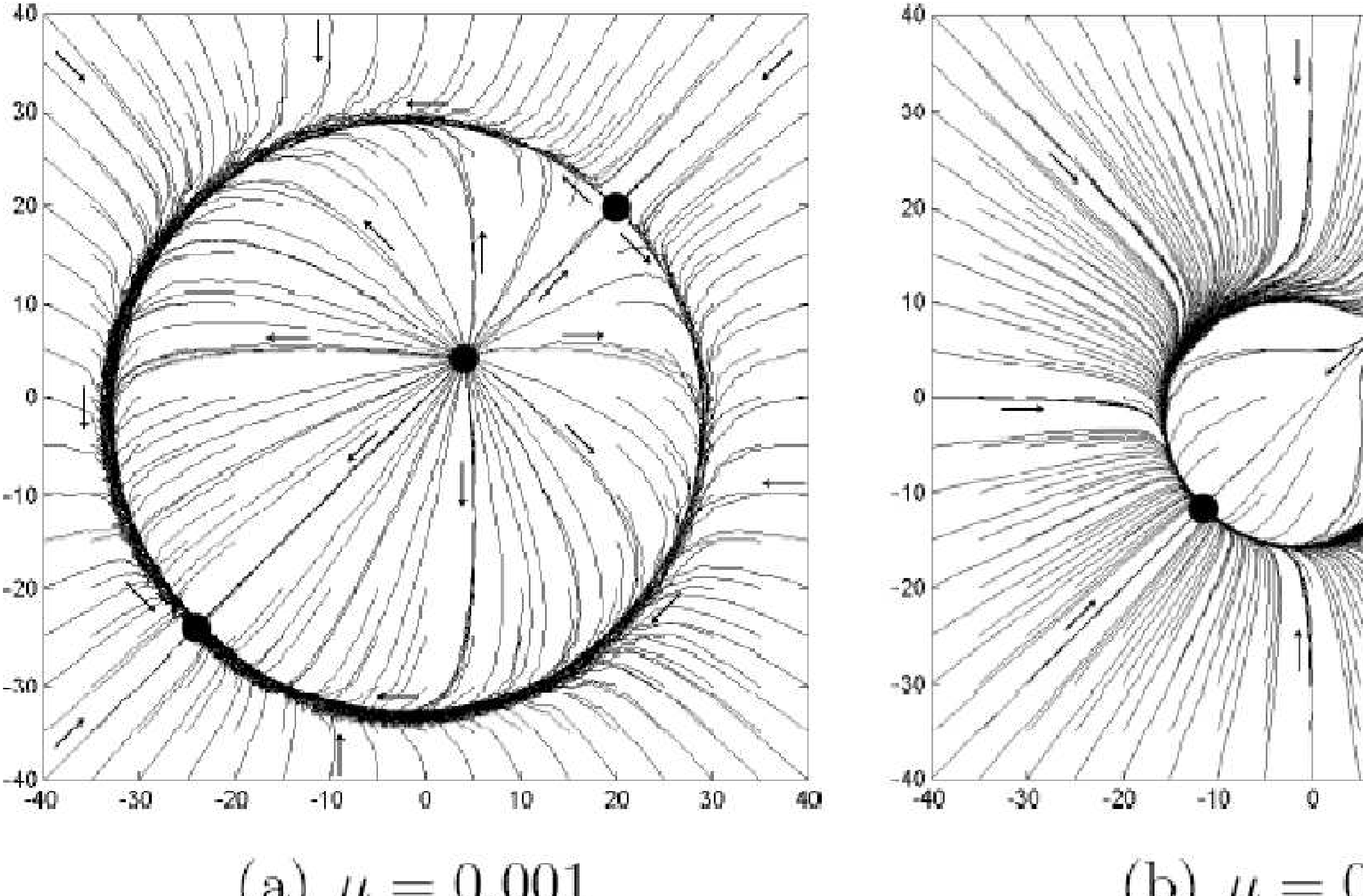

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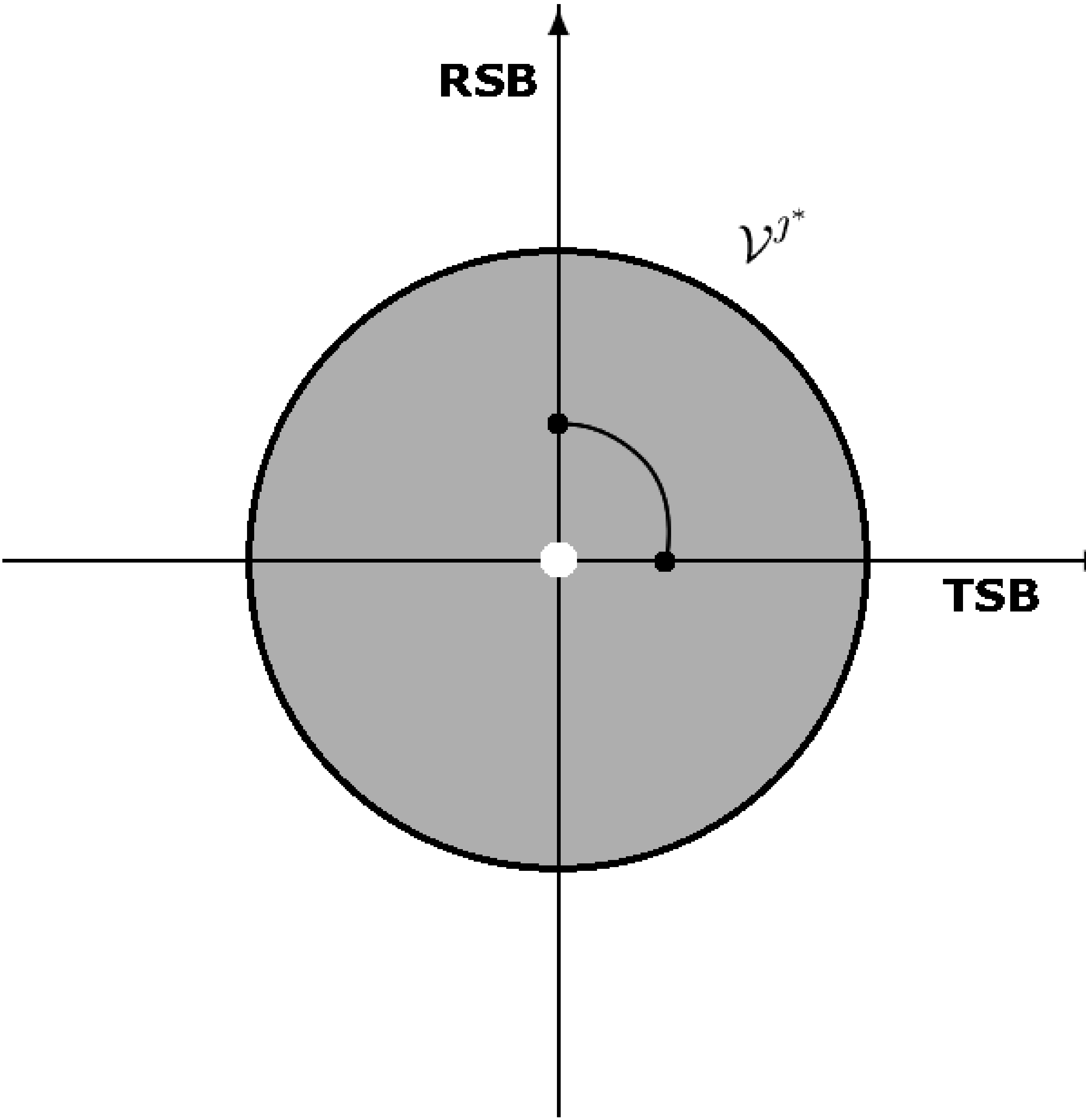
(b) $\mu = 0$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